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Bounded rationality leads to equilibrium of public goods games

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In this work, we introduce a degree of rationality to the public goods games in which players can determine whether or not to participate, and with it a new mechanism has been established. Existence of the bounded rationality would lead to a new equilibrium which differs from the Nash equilibrium and qualitatively explains the fundamental role of loners' payoff for maintaining cooperation. Meanwhile, it is shown how the potential strategy influences the players' decision. Finally, we explicitly demonstrate a rock-scissors-paper dynamics which is a consequence of this model.

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I. INTRODUCTION

The public goods games (PGGs) can be regarded as a very useful tool to qualitatively investigate interactions among human beings [1–4]. For biologists, ecologists, economists, and social scientists it has been a long history of employing the PGG as a model to study how to maintain cooperation in a group of anonymous individuals [5–9]. It can be traced back to the tragedy of the commons [10,11]. In that model, each individual confronts a temptation of making null contributions, but exploiting other cooperators.

In a typical PGG, N players are randomly selected from a large population. In the game, there are three different behavioral types of players: (a) the cooperators C who are ready to participate in the PGG and willing to contribute a fixed amount of money c to the common pool, (b) the defectors D who participate but refuse to pay and attempt to exploit the resource of the common pool, and (c) the loners L who are unwilling to participate in the PGG, but gain a fixed payoff Kc which is a small amount of money. A practical co-existence of these strategies leads to a rock-scissors-paper dynamics with a cyclic dominance [12].

Recently, the PGG has been respectively studied from theoretical and experimental aspects [2,12–16]. One of the available theories is the replicator dynamics which predicts the frequency distributions of the three strategies and corresponding payoffs at next step by the information of the present state [2,17]. But there is an obvious flaw in this theory, which when the frequency of any of the three strategies is zero, the dynamics fails to function forever. To verify the replicator dynamics, some experiments were conducted in 2003 [15]. The experiments demonstrate that qualitatively a rock-scissors-paper dynamics prevails; however, the quantitative results (including frequencies and average payoffs of the three strategies) are different from the predictions of Ref. [2]. It means that the model given in Ref. [2] does not fully suit to the situation under discussion.

In this work, we introduce the degree of rationality ϵ with which a new mechanism is established to achieve equilibrium. In fact, this mechanism is a refinement to the original (noncooperative) game theory [1,18], (see details in Ref. [19]). By this means, one can change the next-step strategy according to the gained information and eventually reach the final equilibrium. Our results are basically the same as that given in Ref. [15], but in our scheme the disappeared strategy may be revived.

II. MODEL AND DYNAMICS

In this model, we consider a simple case that N players are randomly selected from a large population. The players who voluntarily participate in the PGG consist of cooperators C, defectors D, and loners L. n_c, n_d, and n_l (with $n_c+n_d+n_l=N$) are the numbers of players who choose to be C, D, and L, respectively, and the net payoffs of the cooperators, defectors, and loners are P_c, P_d, and P_l whose explicit expressions are given below,

$$P_c = \frac{rcn_c}{n_c + n_d} - c, \tag{1}$$

$$P_d = \frac{rcn_c}{n_c + n_d}, \tag{2}$$

$$P_l = Kc, \tag{3}$$

where r denotes the interest rate of the common pool. In particular, if only one player participates in the PGG (i.e., $n_c+n_d=1$), he should be accounted as a loner. Obviously, when $K \le 0$, the loner will not obtain any positive payoff (namely, he cannot do any better than a defector); when $r \le 1+K$, no matter what a cooperator does, he cannot make more money than a loner; whereas when $r \ge N$, no matter

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what a defector does, he cannot prevail a cooperator. Therefore for the PGG in which players voluntarily participate, there is a constraint,

$$1 < 1 + K < r < N. \tag{4}$$

The above expression indicates that there is not a dominating pure strategy in voluntary PGG, and this also leads to the rock-scissors-paper cycling dynamics of the three various strategies in the system. If there are too many cooperators, defectors may gain higher payoffs and the consequence would be a growth of the number of defectors; whereas if the number of defectors is too large, the payoffs of cooperators would become very low, and this would increase the number of loners; if the number of loners is too large, they would gain more payoffs in a group consisting of not many participants, so more players would choose to be cooperators. That is to say, as the constraint (4) is satisfied, three strategies can coexist.

We let x_c, x_d, and x_l denote the frequencies of cooperators, defectors, and loners, respectively (with the normalization condition $x_c + x_d + x_l = 1$), then their average payoff values are [2,14]

$$P_d = Kcx_l^{N-1} + rc\frac{x_c}{1-x_l}\left(1 - \frac{1-x_l^N}{N(1-x_l)}\right), \tag{5}$$

$$P_c = P_d - c - c(r-1)x_l^{N-1} + c\frac{r}{N}\frac{1-x_l^N}{1-x_l}. \tag{6}$$

According to Nash's noncooperative game theory [18], the Nash equilibrium (NE) condition of this game should be

$$P_c = P_d = P_l = Kc. \tag{7}$$

As we take the parameters provided by Ref. [15]: $N=6$, $r=3.6$, $c=1.25$, and $K=1$, the NE should be $(x_l, x_c, x_d) = (0.4240, 0.2215, 0.3545)$. In fact, it is also the fixed point in Ref. [2]. However, the NE is significantly different from the results of experiments that were carried out in 2003 [15]. This explicitly indicates that in realistic life, human beings are not perfectly rational. Besides, by Eqs. (5) and (6) one notices that when $r \leq 2$, cooperators by no means obtain a higher payoff than defectors; therefore as long as the players are perfectly rational, only the interest rate satisfies the condition $\mathrm{Max}(1+K, 2) < r < N$, and all the three strategies can coexist.

In this work, we introduce a degree of rationality representing the maximal difference of the payoffs between various strategies that the players in voluntary PGG can accept. We postulate that the payoff difference players can accept is proportional to the following: (i) the difference between the maximal payoff for cooperators and that for loners; (ii) the maximal increase of payoff a cooperator receives when he or she switches from cooperation to defection. For simplicity and without loss of generality, we set the proportional coefficient equal to one; thus the degree of rationality (i.e., the maximal difference of payoffs players can accept) can be written as

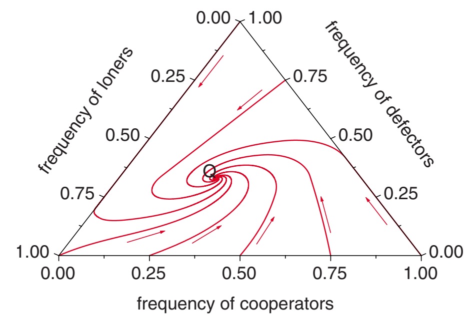

FIG. 1. (Color online) The dynamics of Eq. (9) for $N=5$, $r=3$, and $K=c=1$. The lines start in various points but all end up at the same fixed point Q. The arrows indicate their directions of evolution.

$$\epsilon = \frac{c(r-1-K)(N-r)}{N}, \tag{8}$$

and the evolution of the frequency x_i of the i strategy is set as [18,19]

$$\dot{x}_i = \frac{c_i(\mathbf{x})}{\sum_j c_j(\mathbf{x})} - x_i, \tag{9}$$

where

$$c_i(\mathbf{x}) = \max[0, P_i(\mathbf{x}) - \max_j(P_j(\mathbf{x})) + \epsilon], \tag{10}$$

with $\mathbf{x} = (x_l, x_c, x_d)$ denoting the strategy profile. It is noted that because both $P_i(\mathbf{x})$ and ϵ contain a factor c; thus Eq. (9) is unrelated to the consumption of the cooperator c.

Let $\dot{\mathbf{x}} = 0$, i.e., the relative frequencies of all the strategies of the players are unchanged, by Eq. (9), one can obtain

$$x_i = \frac{c_i(\mathbf{x})}{\sum_j c_j(\mathbf{x})}, \tag{11}$$

where $\mathbf{x}$ denotes an equilibrium of the strategies with the bounded rationality. Obviously, the equilibrium of the system is also unrelated to c, thus for simplification we can set c to be unity without losing generality.

III. RESULTS AND DISCUSSION

Now, let us start analyzing the dynamical behaviors of the system in our model. Figure 1 shows that our dynamical equation would drive the system to finish at a fixed point Q; meanwhile it is also shown: (i) if the loners are prevalent, the number of cooperators would increase, (ii) if most of the players have chosen to cooperate, more players then would turn to defect, and (iii) if most of the players are defectors, then the best choice for the majority of players would be leaving the game to be loners. Although previous theories provided a rock-scissors-paper cycle [14], such a trend is obviously observed in our model. This result is consistent with experimental result [15].

In order to qualitatively analyze the differences of equilibrium for players with bounded rationality and that for the

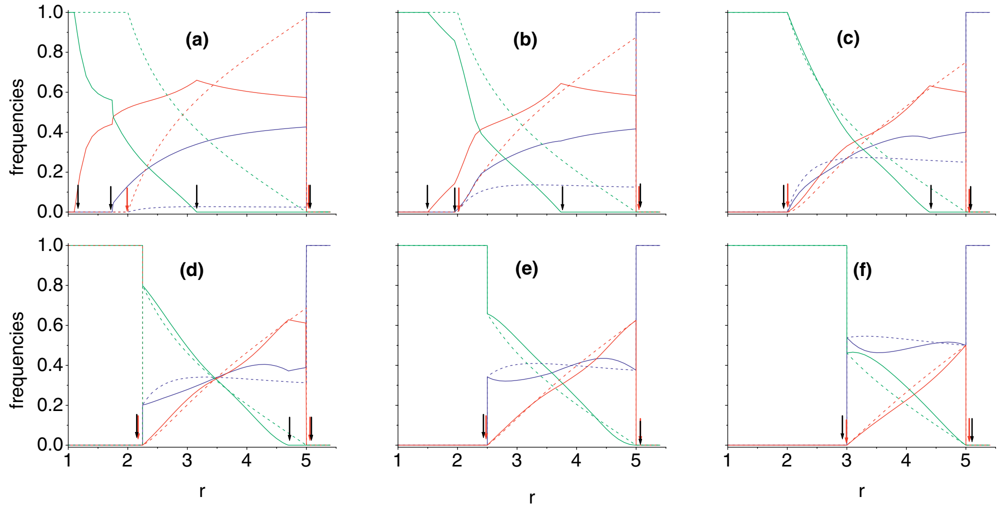

FIG. 2. (Color) Average frequencies of cooperators C (blue line), defectors D (red line), and loners L (green line) as a function of r for parameters $N=5$ and (a) $K=0.1$, (b) $K=0.5$, (c) $K=1.0$, (d) $K=1.25$, (e) $K=1.5$, and (f) $K=2.0$. The NE is shown as blue dashed line (c), red dashed line (d), and green dashed line (l). The black arrows indicate where strategies burst or vanish in one model and the red arrows indicate where strategies burst or vanish in Nash's theory.

perfectly rational ones (NE), it would be helpful to study the information gained from Fig. 2. For convenience of discussion, we define x_i and $\hat{x}_i(i=c,d,l)$ as the frequencies of the i strategy in our model and Nash's theory, respectively.

By the Nash's theory, all players are perfectly rational, so it implies that all the three strategies coexist in the range of $\mathrm{Max}(1+K,2)<r<N$ and within the coexistence range the frequency of loners is unrelated to their payoffs. However, as the rationality degree of the players is bounded (i.e., bounded rationality), the coexistence range of the three strategies would obviously shift and the frequency of loners would be affected by both r and K.

Figure 2 exhibits that as the payoff of loners increases (K becomes larger), the players with bounded rationality more likely choose to be loners than those with perfect rationality. It means that with increase in the payoff of the loners, x_l would gradually increase from smaller than $\hat{x}_l$ to be greater than $\hat{x}_l$. In order to qualitatively describe the effect of K on the equilibrium, we have calculated the dependence of the extinction points of cooperators and loners (r_C, r_L) on K as we only change income of loners but keeping N unchanged, in the bounded rationality model (see Fig. 3). Obviously, the loners would vanish earlier and earlier as K diminishes.

Figures 2(c)–2(e) exhibit that x_l gradually decreases from greater than $\hat{x}_l$ to smaller as the interest rate r increases. This indicates that players with bounded rationality more favor taking a risk when the payoff of the cooperators increases (i.e., r is getting larger) than those with perfect rationality. It implies that the loners with bounded rationality would be eliminated much earlier than those with perfect rationality. For example, as $N=5$, $r=3$, and $K=c=1$, the loners with bounded rationality would vanish at $r_L=4.4$ (the extinction point is consistent with the results of Refs. [14,12]). In Nash's theory, instead, in any case loners would vanish at $r=N$. Here, r_α refers to the extinction point of the α strategy in the bounded rationality model.

By Eqs. (1) and (2), it is easy to obtain that when a participant switches from cooperation to defection, the enhancement of his payoff would be $c-rc/(n_c+n_d)$ [2]. It implies that as r increases, because the gain of payoff which a cooperator receives as he turns into a defector decreases, the temptation of choosing defection decreases accordingly. As r is sufficiently large, the participants with bounded rationality would more likely choose cooperation than those with perfect rationality ($x_c>\hat{x}_c$).

From Fig. 2, one can obtain some additional pieces of information: (1) as $r-1-K$ is sufficiently large, the cooperators can survive, and even though as $r<2$ they still do. (2) as $r>r_L$, x_d decreases and x_c increases as r increases; this consequence is consistent with the results of Refs. [12,14]. Besides, even though in the case of $r>r_L$, loners disappear, the payoff of loners still seriously affects the choice of the participants; (3) under the condition of bounded rationality, the range for defectors' survival is $1+K<r<N$. As both r and K are sufficiently small, all cooperators are extinguished, but defectors still survive. This indicates that the cooperators who possibly survive (if everybody selects cooperation, being a cooperator may still be a possibility to gain positive payoffs greater than the loners' payoff) can exert important influence on the selection trend of participants; even though

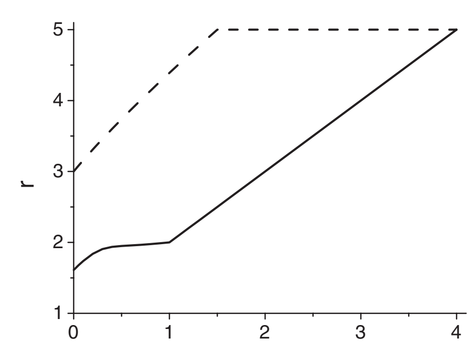

FIG. 3. The values of r_C (solid line) and r_L (dashed line), where cooperators burst and loners vanish, respectively, as a function of K with $N=5$.

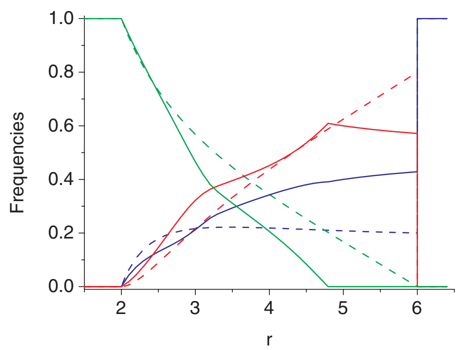

FIG. 4. (Color) Average frequencies of cooperators *C* (blue line), defectors *D* (red line), and loners *L* (green line) as a function of *r* for parameters $N=6$, $K=1$, and $c=1.25$. The NE is shown as blue dashed line (c), red dashed line (d), and green dashed line (l). The black arrows indicate where strategies burst or vanish in one model and the red arrows indicate where strategies burst or vanish in Nash's theory.

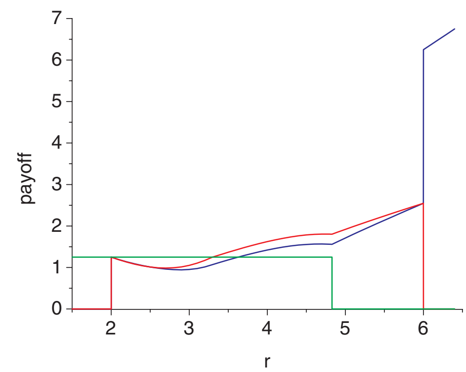

FIG. 5. (Color) Average payoffs of cooperators *C* (blue line), defectors *D* (red line), and loners *L* (green line) as a function of *r* for parameters $N=6$, $K=1$, and $c=1.25$.

in practice, cooperators are already extinguished. Here, we define a strategy which has already disappeared, but still possesses a possibility to bring up more payoffs than the other strategies as a potential strategy, and this potential strategy would seriously affect the equilibrium of the system.

Finally, let us analyze the average payoffs of various strategies in the bounded rationality model. In fact, by Eq. (10), it is easy to observe that the frequencies of various strategies and their average payoffs are disposed in the same orders. For example, when the frequency of the defectors prevails, their average payoff must be the highest. With this understanding, we can learn the payoff distribution of various strategies via simply analyzing Fig. 2. In order to make a clear comparison of the results of our model with that given in Ref. [15], it would be helpful to plot the frequencies-*r* and payoff-*r* diagrams with parameter set as $N=6$, $K=1$, and $c=1.25$ (See Figs. 4 and 5).

One can easily observe in Fig. 5 that within a rather large range, the average payoff of defectors is higher than that of cooperators, and definitely when *r* is small, the payoff of cooperators is even lower than that of loners. All this manifests that as *r* is small, it is wise not to participate in the game; by contrary, only as *r* is sufficiently large, participation in the game can bring up more payoff than loners, and moreover, the defectors always gain more payoff than others. The experimental results show that as $r=3.6$, the average payoff of defectors is the highest (1.46 ± 0.04 Eu.), and that of cooperators is the second (1.32 ± 0.09 Eu.) [15]. Our theoretical calculation predicts that the average payoff of defectors is the highest (1.48 Eu.), and that of cooperators takes the second place (1.28 Eu.). This result is comparatively consistent with the experimental data. Besides, the prediction of our theory on the frequencies of various strategies ($x_l=29.0\%$, $x_c=30.4\%$, $x_d=40.6\%$) is also closer to the experimental measurements.

Our results indicate that in the future when one investigates the human cooperation evolution, he must especially concern the human bounded rationality behavior.

IV. SUMMARY

In this work, we introduce the degree of human rationality in the anonymous public goods games (PGG) and then establish a new dynamical equation. In this model, people can select to participate in the PGG as a cooperator or a defector, or not to participate in the game as a loner. Our result shows that the payoff of loners plays an extremely important role to the equilibrium of the system. When the payoff of loners is relatively low, people are tempted to participate in PGG; whereas as the payoff of loners is comparatively high, people select to escape from PGG. This leads to the consequence that loners would be extinguished when the interest rate is high. Besides, our work also indicates that (1) the potential strategy still seriously affects the equilibrium of the system; (2) no matter what state the system resides in at the initial moment, our dynamical equation would drive the system to a stable equilibrium in the rock-scissors-paper way.

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